A Cut-Free Sequential System for the Propositional Modal Logic of Finite Chains

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A Cut-Free Sequential System for the Propositional Modal Logic of Finite Chains

By

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Abstract

The main purpose of this paper is to give a cut-free Gentzen-type sequential system for **K4.3G** of finite chains. The cut-elimination theorem is proved both model-theoretically and proof-theoretically.

§ 1. Introduction

There are thousands of modal logics, only a bit of which enjoy Gentzen-type sequential formulations. Modal logics with cut-free sequential systems are even fewer and it is often a challenging problem to find out such pleasant formulations to a given modal logic. See Zeman [7] for the general reference and Sato [5] for an example of recent such attempts. The main purpose of this paper is to give a cut-free sequential system for **K4.3G** of Gabbay [2, § 25].

Formulas (of **K4.3G**) are constructed from propositional variables p and $\perp$ (falsity) by using $\supset$ (implication) and $\Box$ (necessity). Other connectives like $\wedge$ (conjunction), $\vee$ (disjunction) and $\neg$ (negation) can be introduced as defined symbols in the usual manner. A *structure* (for **K4.3G**) is a quadruple (S, R, s_0, D_s), where

(1) S is a nonempty finite set.
(2) R is an irreflexive transitive binary relation on S such that either xRy or yRx for any distinct $x, y \in S$.
(3) $s_0 \in S$.
(4) For any $s \in S$, D_s assigns a truth-value 0 or 1 to every propositional variable.

Given a structure (S, R, s_0, D_s), the truth-value $\|A\|_s$ of a formula A at $s \in S$ is defined inductively as follows:

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(1) $\|p\|_s = D_s(p)$ for any propositional variable p.
(2) $\|\bot\|_s = 0$.
(3) $\|A \supset B\|_s = 1$ iff $\|A\|_s = 0$ or $\|B\|_s = 1$.
(4) $\|\Box A\|_s = 1$ iff for any $t \in S$ such that sRt, $\|A\|_t = 1$.

If $\|A\|_{s_0} = 1$ for any structure (S, R, s_0, D_s), then A is called *valid*, notation: $\models A$.

K4.3G can be axiomatized by the classical propositional calculus plus the following axioms and inference rules.

(A1) $\Box(A \supset B) \supset (\Box A \supset \Box B)$
(A2) $\Box A \supset \Box\Box A$
(A3) $\Box(\Box A \supset A) \supset \Box A$
(A4) $\Box(\Box A \supset B) \vee \Box(B \wedge \Box B \supset A)$
(R1) $\dfrac{A}{\Box A}$

We write $\vdash_{\mathbf{K4.3G}} A$ if A is provable in the above formal system.

Theorem 1.1. *For any formula A, $\vdash_{\mathbf{K4.3G}} A$ iff $\models A$.*

In the next section we present our sequential system **SK4.3G** and establish its cut-freeness semantically while its purely syntactic proof is given in Section 3. Finally we admit that this paper was inspired by a cut-free sequential system of Leivant [4] for the modal logic **K4G** of finite partial orders but its subtle error in the proof of the cut-elimination theorem is corrected in our more general context.

§ 2. Cut-free System for K4.3G

A *sequent* is an ordered pair (Γ, Δ) of (possibly empty) finite sets of formulas, which we usually denote by $\Gamma \to \Delta$. We use such self-explanatory notations as $A, \Gamma \to \Delta, B$ for $\{A\} \cup \Gamma \to \Delta \cup \{B\}$ and $\Box\Gamma$ for $\{\Box A : A \in \Gamma\}$ freely.

Our sequential formal system **SK4.3G** ("S" for "Sequential") consists of the following axioms and inference rules:

Axioms: $A \to A$

$\bot \to$

Rules: $\dfrac{\Gamma \to \Delta}{\Pi, \Gamma \to \Delta, \Lambda}$ (thin)

$\dfrac{\Gamma \to \Delta, A \quad B, \Pi \to \Lambda}{A \supset B, \Gamma, \Pi \to \Delta, \Lambda}$ ($\supset$L)

$\dfrac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B}$ ($\supset$R)

$$\frac{\{\Gamma,\ \Box\Gamma,\ \Box\Pi\to\Pi,\ \Box\Lambda:\ \Pi\cup\Lambda=\Delta,\ \Pi\cap\Lambda=\emptyset \text{ and } \Pi\neq\emptyset\}}{\Box\Gamma\to\Box\Delta}\ (\mathrm{GL4.3})$$

, where $\Delta\neq\emptyset$ in (GL4.3).

It is easy to show that the following rules are admissible in **SK4.3G**.

$$\frac{\Gamma\to\Delta,\ A}{\neg A,\ \Gamma\to\Delta}\ (\neg\mathrm{L})$$

$$\frac{A,\ \Gamma\to\Delta}{\Gamma\to\Delta,\ \neg A}\ (\neg\mathrm{R})$$

$$\left.\begin{array}{c}\dfrac{A,\ \Gamma\to\Delta}{A\wedge B,\ \Gamma\to\Delta}\\[2ex] \dfrac{B,\ \Gamma\to\Delta}{A\wedge B,\ \Gamma\to\Delta}\end{array}\right\}\ (\wedge\mathrm{L})$$

$$\frac{\Gamma\to\Delta,\ A\quad \Gamma\to\Delta,\ B}{\Gamma\to\Delta,\ A\wedge B}\ (\wedge\mathrm{R})$$

$$\frac{A,\ \Gamma\to\Delta\quad B,\ \Gamma\to\Delta}{A\vee B,\ \Gamma\to\Delta}\ (\vee\mathrm{L})$$

$$\left.\begin{array}{c}\dfrac{\Gamma\to\Delta,\ A}{\Gamma\to\Delta,\ A\vee B}\\[2ex] \dfrac{\Gamma\to\Delta,\ B}{\Gamma\to\Delta,\ A\vee B}\end{array}\right\}\ (\vee\mathrm{R})$$

If $\Gamma\to\Delta$ is provable in **SK4.3G**, we write $\vdash_{\mathbf{SK4.3G}}\Gamma\to\Delta$. We notice that the rule (GL4.3) has the variable number of upper sequents, depending on the number $|\Delta|$. If $|\Delta|=1$, our rule (GL4.3) degenerates into the rule (GL) of Leivant [4].

$$\frac{\Gamma,\ \Box\Gamma,\ \Box A\to A}{\Box\Gamma\to\Box A}\ (\mathrm{GL})$$

If $|\Delta|=2$, then the rule (GL4.3) goes as follows:

$$\frac{\Gamma,\ \Box\Gamma,\ \Box A,\ \Box B\to A,\ B\qquad \Gamma,\ \Box\Gamma,\ \Box A\to A,\ \Box B\qquad \Gamma,\ \Box\Gamma,\ \Box B\to\Box A,\ B}{\Box\Gamma\to\Box A,\ \Box B}$$

To deepen the reader's understanding of the rule (GL4.3), we shall show that $\vdash_{\mathbf{SK4.3G}}\to\Box(\Box A\supset B)\vee\Box(B\wedge\Box B\supset A)$.

We have the following proof π_1 of the sequent $\Box(\Box A\supset B),\ \Box(B\wedge\Box B\supset A)\to\Box A\supset B,\ B\wedge\Box B\supset A$.

$$\dfrac{\dfrac{\dfrac{\dfrac{B\to B}{\Box(\Box A\supset B),\ \Box(B\wedge\Box B\supset A),\ \Box A,\ B\to A,\ B}\ (\mathrm{thin})}{\Box(\Box A\supset B),\ \Box(B\wedge\Box B\supset A),\ \Box A,\ B\wedge\Box B\to A,\ B}\ (\wedge\mathrm{L})}{\Box(\Box A\supset B),\ \Box(B\wedge\Box B\supset A),\ \Box A\to B,\ B\wedge\Box B\supset A}\ (\supset\mathrm{R})}{\Box(\Box A\supset B),\ \Box(B\wedge\Box B\supset A)\to\Box A\supset B,\ B\wedge\Box B\supset A}\ (\supset\mathrm{R})$$

We have the following proof π_2 of the sequent $\Box(\Box A\supset B)\to\Box A\supset B,\ \Box(B\wedge\Box B\supset A)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A\to A}{\Box(\Box A\supset B),\ \Box A\supset B,\ \Box A,\ A,\ \Box(B\wedge\Box B\supset A),\ B\wedge\Box B\to A}\ (\text{thin})}{\Box(\Box A\supset B),\ \Box A\supset B,\ \Box A,\ A,\ \Box(B\wedge\Box B\supset A)\to B\wedge\Box B\supset A}\ (\supset\text{R})}{\Box(\Box A\supset B),\ \Box A\to\Box(B\wedge\Box B\supset A)}\ (\text{GL4.3})}{\Box(\Box A\supset B),\ \Box A\to B,\ \Box(B\wedge\Box B\supset A)}\ (\text{thin})}{\Box(\Box A\supset B)\to\Box A\supset B,\ \Box(B\wedge\Box B\supset A)}\ (\supset\text{R})$$

We have the following proof π_3 of the sequent $\Box(B\wedge\Box B\supset A)\to\Box(\Box A\supset B),\ B\wedge\Box B\supset A$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B\to B}{\Box(B\wedge\Box B\supset A),\ B\wedge\Box B\supset A,\ \Box B,\ B,\ \Box(\Box A\supset B),\ \Box A\to B}\ (\text{thin})}{\Box(B\wedge\Box B\supset A),\ B\wedge\Box B\supset A,\ \Box B,\ B,\ \Box(\Box A\supset B)\to\Box A\supset B}\ (\supset\text{R})}{\Box(B\wedge\Box B\supset A),\ \Box B\to\Box(\Box A\supset B)}\ (\text{GL4.3})}{\Box(B\wedge\Box B\supset A),\ \Box B\to\Box(\Box A\supset B),\ A}\ (\text{thin})}{\Box(B\wedge\Box B\supset A),\ B\wedge\Box B\to\Box(\Box A\supset B),\ A}\ (\wedge\text{L})}{\Box(B\wedge\Box B\supset A)\to\Box(\Box A\supset B),\ B\wedge\Box B\supset A}\ (\supset\text{R})$$

Therefore

$$\dfrac{\dfrac{\vdots\,\}\pi_1 \qquad \vdots\,\}\pi_2 \qquad \vdots\,\}\pi_3}{\to\Box(\Box A\supset B),\ \Box(B\wedge\Box B\supset A)}\ (\text{GL4.3})}{\to\Box(\Box A\supset B)\vee\Box(B\wedge\Box B\supset A)}\ (\vee\text{R})$$

A sequent $\Gamma\to\Delta$ is called *realizable* if for some structure (S, R, s_0, D_s), $\|A\|_{s_0}=1$ for any $A\in\Gamma$ and $\|B\|_{s_0}=0$ for any $B\in\Delta$. A sequent $\Gamma\to\Delta$ which is not realizable is called *valid,* notation: $\models\Gamma\to\Delta$.

Theorem 2.1. (Soundness Theorem). *For any sequent $\Gamma\to\Delta$, if $\vdash_{\mathbf{SK4.3G}}\Gamma\to\Delta$, then $\models\Gamma\to\Delta$.*

Proof. By induction on a proof of $\Gamma\to\Delta$.

Corollary 2.2. (Consistency). *The empty sequent $\to$ is not provable in* **SK4.3G**.

Theorem 2.3. (Completeness Theorem). *For any sequent $\Gamma\to\Delta$, if $\models\Gamma\to\Delta$, then $\vdash_{\mathbf{SK4.3G}}\Gamma\to\Delta$.*

Proof. Let $\Gamma\to\Delta$ be the given sequent. We denote by Ω the set of all subformulas occurring in a formula of $\Gamma\cup\Delta$. A sequent $\Pi\to\Lambda$ is called Ω-*saturated* if it satisfies the following conditions:

(1) $\nvdash_{\mathbf{SK4.3G}}\Pi\to\Lambda$.

(2) $\Pi\cup\Lambda\subseteq\Omega$.

(3) For any $A \in \Omega-(\Pi \cup \Lambda)$, $\vdash_{\mathbf{SK4.3G}} \Pi \to \Lambda, A$ and $\vdash_{\mathbf{SK4.3G}} A, \Pi \to \Lambda$.

Assuming that $\nvdash_{\mathbf{SK4.3G}} \Gamma \to \Delta$, we shall show that $\not\models \Gamma \to \Delta$. We denote by $W(\Omega)$ the set of all Ω-saturated sequents. Since $\nvdash_{\mathbf{SK4.3G}} \Gamma \to \Delta$, the sequent $\Gamma \to \Delta$ can be extended to some $\Gamma_0 \to \Delta_0 \in W(\Omega)$. For any set Σ of formulas, $(\Sigma)_\Box$ denotes the set of all formulas A such that $\Box A \in \Sigma$. If $(\Delta_0)_\Box = \emptyset$, then let $S = \{\Gamma_0 \to \Delta_0\}$. If $(\Delta_0) \neq \emptyset$, $\nvdash_{\mathbf{SK4.3G}} \Box(\Gamma_0)_\Box \to \Box(\Delta_0)_\Box$. Therefore, taking the rule (GL4.3) into consideration, there exist two sets Σ_1, Σ_2 of formulas such that:

(1) $\Sigma_1 \neq \emptyset$.
(2) $\Sigma_1 \cup \Sigma_2 = (\Delta_0)_\Box$.
(3) $\Sigma_1 \cap \Sigma_2 = \emptyset$.
(4) $\nvdash_{\mathbf{SK4.3G}} (\Gamma_0)_\Box, \Box(\Gamma_0)_\Box, \Box\Sigma_1 \to \Sigma_1, \Box\Sigma_2$.

The sequent $(\Gamma_0)_\Box, \Box(\Gamma_0)_\Box, \Box\Sigma_1 \to \Sigma_1, \Box\Sigma_2$ can be extended to some $\Gamma_1 \to \Delta_1 \in W(\Omega)$. We notice that:

(1) $(\Gamma_0)_\Box \subset (\Gamma_1)_\Box$ (By $\subset$ we denote the proper inclusion).
(3) $(\Gamma_0)_\Box \subseteqq \Gamma_1$.
(3) $(\Delta_0)_\Box \subseteqq \Delta_1 \cup (\Delta_1)_\Box$.

If $(\Delta_1)_\Box = \emptyset$, then we let $S = \{\Gamma_0 \to \Delta_0, \Gamma_1 \to \Delta_1\}$. If $(\Delta_1)_\Box \neq \emptyset$, we repeat the above process. In any case we finally obtain a sequence $\{\Gamma_i \to \Delta_i\}_{i=0}^{k}$ of $W(\Omega)$ such that:

(1) $(\Gamma_0)_\Box \subset (\Gamma_1)_\Box \subset \cdots \subset (\Gamma_k)_\Box$.
(2) $(\Gamma_i)_\Box \subseteqq \Gamma_{i+1}$ for any $0 \leqq i < k$.
(3) $(\Delta_i)_\Box \subseteqq \Delta_{i+1} \cup (\Delta_{i+1})_\Box$ for any $0 \leqq i \leqq k-1$.
(4) $(\Delta_i)_\Box \neq \emptyset$ for any $i < k$ and $(\Delta_k)_\Box = \emptyset$.

Set $S = \{\Gamma_0 \to \Delta_0, \cdots, \Gamma_k \to \Delta_k\}$. We let $(\Gamma_i \to \Delta_i) R (\Gamma_j \to \Delta_j)$ iff $i < j$. Let $s_0 = \Gamma_0 \to \Delta_0$. We define $D_{\Gamma_i \to \Delta_i}$ as follows:

$D_{\Gamma_i \to \Delta_i}(p) = 1$ iff $p \in \Gamma_i$ for any propositional variable p.

It is not difficult to show by induction on $A \in \Omega$ that for any $0 \leqq i \leqq k$,

(1) $\|A\|_{\Gamma_i \to \Delta_i} = 1$ if $A \in \Gamma_i$.
(2) $\|A\|_{\Gamma_i \to \Delta_i} = 0$ if $A \in \Delta_i$.

In particular, we can conclude that $\Gamma_0 \to \Delta_0$ is realizable and so is $\Gamma \to \Delta$. This completes the proof.

Corollary 2.4. *For any formula A, $\vdash_{\mathbf{K4.3G}} A$ iff $\vdash_{\mathbf{SK4.3G}} \to A$.*

Corollary 2.5. *The following inference rule is admissible in* **SK4.3G.**

$$\frac{\Gamma \to \Delta, A \quad A, \Pi \to \Lambda}{\Gamma, \Pi \to \Delta, \Lambda} \ \text{(cut)}$$

In the next section we will give a purely syntactical proof of Corollary 2.5.

§ 3. Cut-Elimination Theorem

The main purpose of this section is to give a proof-theoretical proof of Corollary 2.5 by amending Gentzen's original proof (for **LK** and **LJ**) such as seen in Takeuti [6].

Theorem 3.1. (Cut-elimination Theorem). *The following inference is admissible in* **SK4.3G.**

$$\frac{\Gamma_1 \to \Delta_1, A \quad A, \Gamma_2 \to \Delta_2}{\Gamma \to \Delta} \text{ (cut)}$$

$$(\Gamma = \Gamma_1 \cup \Gamma_2 \textit{ and } \Delta = \Delta_1 \cup \Delta_2)$$

For technical reasons we deal with a slightly modified version of **SK4.3G**, say **SK4.3G′**, which is obtainable from SK4.3G by restricting rules (thin) and ($\supset$L) to the following (thin)$_\Box$ and ($\supset$L)′

$$\frac{\Box(\Gamma)_\Box \to \Box(\Delta)_\Box}{\Gamma \to \Delta} \text{ (thin)}_\Box$$

$$\frac{\Gamma \to \Delta, A \quad B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta} \text{ }(\supset\text{L})'$$

and instead adopting as axioms sequents $\Gamma \to \Delta$ satisfying at least one of the following conditions:

(1) $p \in \Gamma \cap \Delta$ for some propositional variable p.
(2) $\bot \in \Gamma$.

Lemma 3.2. *The following rule* (thin L) *is admissible in* **SK4.3G′**.

$$\frac{\Gamma \to \Delta}{A, \Gamma \to \Delta} \text{ (thin L)}$$

Proof. It is sufficient to deal with proof figures which contain only one (thin L) as the last inference.

$$\begin{array}{c} \ddots \vdots \cdot^{\cdot} \\ \dfrac{\Gamma \to \Delta}{A, \Gamma \to \Delta} \text{ (thin L)} \end{array}$$

The proof is carried out by double induction mainly on the formula A and secondly on the length of longest threads of the proof of $\Gamma \to \Delta$. Here we deal only with a special case of (GL4.3) being the last inference of the proof of $\Gamma \to \Delta$.

$$\begin{array}{c}\vdots\\ \Box\Gamma, \Gamma, B\to B \\ \hline \Box\Gamma\to\Box B \\ \hline A,\ \Box\Gamma\to\Box B\end{array}\quad\begin{array}{l}(\mathrm{GL4.3})\\ (\mathrm{thin\ L})\end{array}$$

If A is not of the form $\Box C$, then (thin L) degenerates into (thin)$_\Box$ which is of course admissible. If A is of the form $\Box C$, the above proof figure is transformed into:

$$\begin{array}{c}\vdots\\ \Box\Gamma, \Gamma, B\to B \\ \hline \Box C,\ \Box\Gamma, \Gamma, \Box B\to B \\ \hline \Box C, C,\ \Box\Gamma, \Gamma, \Box B\to B \\ \hline \Box C,\ \Box\Gamma\to\Box B\end{array}\quad\begin{array}{l}(\mathrm{thin\ L})\\ (\mathrm{thin\ L})\\ (\mathrm{GL4.3})\end{array}$$

Therefore the induction process works well.

Lemma 3.3. *The following rule* (thin R) *is admissible in* **SK4.3G′**.

$$\frac{\Gamma\to\Delta}{\Gamma\to\Delta, A}\ (\mathrm{thin\ R})$$

Proof. It is sufficient to deal with proof figures which contain only one (thin R) as the last inference.

$$\begin{array}{c}\vdots\\ \Gamma\to\Delta \\ \hline \Gamma\to\Delta, A\end{array}\quad(\mathrm{thin\ R})$$

The proof is carried out by double induction mainly on the formula A and secondly on the length of longest threads of the proof of $\Gamma\to\Delta$. Here we deal only with a special case of (GL4.3) being the last inference of the proof of $\Gamma\to\Delta$.

$$\begin{array}{c}\vdots\\ \Box\Gamma, \Gamma, \Box B\to B \\ \hline \Box\Gamma\to\Box B \\ \hline \Box\Gamma\to\Box B, A\end{array}\quad\begin{array}{l}(\mathrm{GL4.3})\\ (\mathrm{thin\ R})\end{array}$$

If A is not of the form $\Box C$, then (thin R) degenerates into (thin)$_\Box$, which is of course admissible. If A is of the form $\Box C$, the above proof figure is transformed into:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Box\Gamma,\Gamma,\Box B\to B}}{\Box\Gamma,\Gamma,\Box B,\Box C\to B}\text{(thin L)}}{\Box\Gamma,\Gamma,\Box B,\Box C\to B,C}\text{(thin R)}\quad\dfrac{\dfrac{\vdots}{\Box\Gamma,\Gamma,\Box B\to B}}{\Box\Gamma,\Gamma,\Box B\to B,\Box C}\text{(thin R)}\quad\dfrac{\dfrac{\dfrac{\vdots}{\Box\Gamma,\Gamma\to\Box B}}{\Box\Gamma,\Gamma,\Box C\to\Box B}\text{(thin L)}}{\Box\Gamma,\Gamma,\Box C\to\Box B,C}\text{(thin R)}$$
$$\overline{\qquad\qquad\qquad\qquad\Box\Gamma\to\Box B,\ \Box C\qquad\qquad\qquad\qquad}\text{(GL4.3)}$$

Therefore the induction process works well.

Lemma 3.4. *The rule* (thin) *is admissible in* **SK4.3G′**.

Proof. Follows readily from Lemmas 3.2 and 3.3.

Lemma 3.5. *For any formula* A, $\vdash_{\mathbf{SK4.3G'}} A\to A$.

Proof. By induction on A. Here we deal only with the case of A being of the form $\Box B$.

$$\dfrac{\dfrac{\dfrac{\vdots}{B\to B}}{\Box B,\ B,\ \Box B\to B}\text{(thin)}}{\Box B\to\Box B}\text{(GL4.3)}$$

Proposition 3.6. *For any sequent* $\Gamma\to\Delta$, $\vdash_{\mathbf{SK4.3G}}\Gamma\to\Delta$ *iff* $\vdash_{\mathbf{SK4.3G'}}\Gamma\to\Delta$.

Proof. (1) If part: Trivial. (2) If only part: Use Lemmas 3.4 and 3.5.

Theorem 37. *The following inference is admissible in* **SK4.3G′**.

$$\dfrac{\Gamma_1\to\Delta_1,\ A\quad A,\ \Gamma_2\to\Delta_2}{\Gamma\to\Delta}\text{(cut)}$$

$$(\Gamma=\Gamma_1\cup\Gamma_2 \text{ and } \Delta=\Delta_1\cup\Delta_2)$$

Proof. It is sufficient to deal with proof figures which contain only one (cut) as the last inference. Thus we must consider the following proof figure π.

$$\dfrac{\Gamma_1\to\Delta_1,\ A\quad A,\ \Gamma_2\to\Delta_2}{\Gamma\to\Delta}\text{(cut)}$$

$$(\Gamma=\Gamma_1\cup\Gamma_2 \text{ and } \Delta=\Delta_1\cup\Delta_2)$$

By the *grade* of a formula B, we mean the number of logical symbols contained in B. By $\gamma(\pi)$ we denote the grade of the cut formula A. By $\delta_l^1(\pi)$ we denote the number of formulas of the form $\Box B$ that occur as subformulas of formulas in $\Gamma_1\cup\Delta_1$[1]. We denote by $\delta_l^2(\pi)$ the number of formulas of the form $\Box B$ that occur in Γ_1[2]. Obviously $\delta_l^1(\pi)\geqq\delta_l^2(\pi)$. We denote by $\delta_l(\pi)$ the number $\delta_l^1(\pi)-\delta_l^2(\pi)$. By $\delta_r^1(\pi)$ we denote the number of formulas of the form $\Box B$ that occur as subformulas in $\Gamma_2 \cup \Delta_2$[3]. We denote by $\delta_r^2(\pi)$ the number of formulas of the form $\Box B$ that occur in Γ_2[4]. We denote by $\delta_r(\pi)$ the number $\delta_r^1(\pi) - \delta_r^2(\pi)$. We decree that $\delta(\pi) = \delta_l(\pi) + \delta_r(\pi)$. By $\rho_l(\pi)$ we denote the number of the longest threads that end with the left upper sequent $\Gamma_1 \to \Delta_1, A$ and contain the cut formula A consecutively. Similarly we denote by $\rho_r(\pi)$ the number of the longest threads that end with the right upper sequent $A, \Gamma_2 \to \Delta_2$ and contain the cut formula A consecutively. By $\rho(\pi)$ we mean the number $\rho_l(\pi) + \rho_r(\pi)$. Now our proof proceeds by triple induction mainly on $\gamma(\pi)$, secondly on $\delta(\pi)$ and thirdly on $\rho(\pi)$. Since our proof is not by the usual double induction on $\gamma(\pi)$ and $\rho(\pi)$, we should be careful enough even in dealing with classical cases.

(1) $\rho(\pi) = 2$: Since in rule (GL4.3) the antecedent of the lower sequent is contained in that of every upper sequent, A can be of the form $\Box B$ only when the right upper sequent $\Box B, \Gamma_2 \to \Delta_2$ is an axiom sequent. In this case $\Gamma \to \Delta$ is also an axiom sequent. Therefore the only nontrivial case we must consider goes as follows:

$$\dfrac{\dfrac{A, \Gamma_1 \to \Delta_1, B}{\Gamma_1 \to \Delta_1, A \supset B}\,(\supset \mathrm{R}) \qquad \dfrac{\Gamma_2 \to \Delta_2, A \quad B, \Gamma_2 \to \Delta_2}{A \supset B, \Gamma_2 \to \Delta_2}\,(\supset \mathrm{L})'}{\Gamma \to \Delta}\,(\mathrm{cut})$$

$(\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Delta = \Delta_1 \cup \Delta_2)$

This proof figure π is transformed into:

$$\dfrac{\dfrac{\Gamma_2 \to \Delta_2, A \quad A, \Gamma_1 \to \Delta_1, B}{\Gamma \to \Delta, B}\,(\mathrm{cut}) \qquad B, \Gamma_2 \to \Delta_2}{\Gamma \to \Delta}\,(\mathrm{cut})$$

Since the grades of A and B are smaller than that of $A \supset B$, the induction process works well.

(2) $\rho(\pi) > 2$: There are several nontrivial cases, which we shall consider case by case in the following:

(2a) π is of the following form:

$$\dfrac{\dfrac{\Gamma_1 \to \Delta_1, B, A \quad C, \Gamma_1 \to \Delta_1, A}{B \supset C, \Gamma_1 \to \Delta_1, A}\,(\supset L) \qquad A, \Gamma_2 \to \Delta_2}{B \supset C, \Gamma \to \Delta}\,(\mathrm{cut})$$

$(\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Delta = \Delta_1 \cup \Delta_2)$

1)-4) Repetition is not counted.

Consider the following proof figure π_1:

$$\cfrac{\begin{matrix}\vdots \\ \Gamma_1 \to \Delta_1,\ B,\ A\end{matrix} \qquad \begin{matrix}\vdots \\ A,\ \Gamma_2 \to \Delta_2\end{matrix}}{\Gamma \to \Delta,\ B}\ \text{(cut)}$$

Since $\gamma(\pi_1)=\gamma(\pi)$, $\delta(\pi_1)\leqq\delta(\pi)$ and $\rho(\pi_1)<\rho(\pi)$, we have a cut-free proof π_1' of the sequent $\Gamma\to\Delta,\ B$ by induction hypothesis.

Consider the following proof figure π_2:

$$\cfrac{\begin{matrix}\vdots \\ C,\ \Gamma_1 \to \Delta_1,\ A\end{matrix} \qquad \begin{matrix}\vdots \\ A,\ \Gamma_2 \to \Delta_2\end{matrix}}{C,\ \Gamma \to \Delta}\ \text{(cut)}$$

Since $\gamma(\pi_2)=\gamma(\pi)$, $\delta(\pi_2)\leqq\delta(\pi)$ and $\rho(\pi_2)<\rho(\pi)$, we have a cut-free proof π_2' of the sequent $C,\ \Gamma\to\Delta$ by induction hypothesis. Therefore

$$\cfrac{\left.\begin{matrix}\vdots \\ \Gamma \to \Delta,\ B\end{matrix}\right\}\pi_1' \qquad \left.\begin{matrix}\vdots \\ C,\ \Gamma \to \Delta\end{matrix}\right\}\pi_2'}{B\supset C,\ \Gamma \to \Delta}\ (\supset\mathrm{L})'$$

The following three cases are treated similarly to (2a).

(2b) $\rho_l(\pi)\geqq 2$ and the last inference of the proof of the left upper sequent of (cut) is ($\supset$R).

(2c) $\rho_r(\pi)\geqq 2$ and the last inference of the proof of the right upper sequent of (cut) is $(\supset\mathrm{L})'$.

(2d) $\rho_r(\pi)\geqq 2$ and the last inference of the proof of the right upper sequent of (cut) is ($\supset$R).

(2e) The last inference of the proofs of both upper sequents of (cut) is (GL4.3):

We deal with the following special case, leaving the general treatment to the reader.

$$\cfrac{\cfrac{\begin{matrix}\vdots \\ \Box,\Gamma_1,\Gamma_1,\Box A,\Box B\to A,B\end{matrix} \quad \begin{matrix}\vdots \\ \Box\Gamma_1,\Gamma_1,\Box A\to A,\Box B\end{matrix} \quad \begin{matrix}\vdots \\ \Box\Gamma_{1,1},\Box B\to A,B\end{matrix}}{\Box\Gamma_1\to\Box A,\ \Box B}\,\text{(GL4.3)} \quad \cfrac{\begin{matrix}\vdots \\ \Box\Gamma_2,\Gamma_2,\Box A,A,\Box C\to C\end{matrix}}{\Box\Gamma_2,\ \Box A\to\Box C}\,\text{(GL4.3)}}{\Box\Gamma\to\Box B,\ \Box C}\,\text{(cut)}$$

($\Gamma=\Gamma_1\cup\Gamma_2$ and $\Delta=\Delta_1\cup\Delta_2$)

Consider the following proof figure π_1:

$$\dfrac{\begin{array}{c}\vdots\\ \Box\Gamma_1,\ \Gamma_1,\ \Box A,\ \Box B\to A,\ B\end{array}\quad\begin{array}{c}\vdots\\ \Box\Gamma_2,\ \Gamma_2,\ \Box A,\ A,\ \Box C\to C\end{array}}{\Box\Gamma,\ \Gamma,\ \Box A,\ \Box B,\ \Box C\to B,\ C}\ (\text{cut})$$

Since the grade of A is smaller than that of $\Box A$, there is a cut-free proof π_1' of $\Box\Gamma, \Gamma, \Box A, \Box B, \Box C\to B, C$ by induction hypothesis.

Consider the following proof figure π_2:

$$\dfrac{\begin{array}{c}\vdots\\ \Box\Gamma_1,\ \Gamma_1\Box B\to\Box A,\ B\end{array}\quad\left.\begin{array}{c}\vdots\\ \Box\Gamma,\ \Gamma,\ \Box A,\ \Box B,\ \Box C\to B,\ C\end{array}\right\}\pi_1'}{\Box\Gamma,\ \Gamma,\ \Box B,\ \Box C\to B,\ C}\ (\text{cut})$$

Since $\gamma(\pi_2)=\gamma(\pi)$ and $\delta(\pi_2)<\delta(\pi)$, there is a cut-free proof π_2' of $\Box\Gamma, \Gamma, \Box B, \Box C\to B, C$ by induction hypothesis.

Consider the following proof figure π_3:

$$\dfrac{\begin{array}{c}\vdots\\ \Box\Gamma_1,\ \Gamma_1,\ \Box B\to\Box A,\ B\end{array}\quad\dfrac{\begin{array}{c}\vdots\\ \Box\Gamma_2,\ \Gamma_2,\ \Box A,\ A,\ \Box C\to C\end{array}}{\Box\Gamma_2,\ \Box A\to\Box C}\ (\text{GL4.3})}{\Box\Gamma_1,\ \Gamma_1,\ \Box\Gamma_2,\ \Box B\to B,\ \Box C}\ (\text{cut})$$

Since $\gamma(\pi_3)=\gamma(\pi)$ and $\delta(\pi_3)<\delta(\pi)$, there is a cut-free proof π_3' of $\Box\Gamma, \Gamma, \Box B\to B, \Box C$ by induction hypothesis and Lemma 3.4.

Consider the following proof figure π_4:

$$\dfrac{\begin{array}{c}\vdots\\ \Box\Gamma_1,\ \Gamma_1,\ \Box A\to A,\ \Box B\end{array}\quad\begin{array}{c}\vdots\\ \Box\Gamma_2,\ \Gamma_2\Box A,\ A,\ \Box C\to C\end{array}}{\Box\Gamma,\ \Gamma,\ \Box A,\ \Box C\to\Box B,\ C}\ (\text{cut})$$

Since $\gamma(\pi_4)<\gamma(\pi)$, there is a cut-free proof π_4' of $\Box\Gamma, \Gamma, \Box A, \Box C\to\Box B, C$ by induction hypothesis.

Consider the following proof figure π_5:

$$\dfrac{\begin{array}{c}\vdots\\ \Box\Gamma_1\to\Box A,\ \Box B\end{array}\quad\left.\begin{array}{c}\vdots\\ \Box\Gamma,\ \Gamma,\ \Box A,\ \Box C\to\Box B,\ C\end{array}\right\}\pi_4'}{\Box\Gamma,\ \Gamma,\ \Box C\to\Box B,\ C}\ (\text{cut})$$

Since $\gamma(\pi_5)=\gamma(\pi)$ and $\delta(\pi_5)<\delta(\pi)$, there is a cut-free proof π_5' of $\Box\Gamma, \Gamma, \Box C\to\Box B, C$ by induction hypothesis. Therefore

$$\dfrac{\begin{matrix}\vdots\\ \Box\Gamma, \Gamma, \Box B, \Box C\to B, C\end{matrix}\Bigg\}\pi_2' \quad \begin{matrix}\vdots\\ \Box\Gamma, \Gamma, \Box B\to B, \Box C\end{matrix}\Bigg\}\pi_3' \quad \begin{matrix}\vdots\\ \Box\Gamma, \Gamma, \Box C\to\Box B, C\end{matrix}\Bigg\}\pi_5'}{\Box\Gamma\to\Box B, \Box C}\,(\mathrm{GL4.3})$$

Before leaving the above proof, the reader should realize that the main reason for dealing with **SK4.3G′** instead of **SK4.3G** directly is to make the secondary induction on $\delta(\pi)$ work well. It seems that the secondary induction of Theorem 3.4 (cut-elimination theorem) of Leivant [4] indeed works well for cases like (a special case of) (2e) but fails to preserve the usual treatment of classical cases like (2c).

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